Let X be a projective variety, and L a line bundle on X.
The <u>section ring</u> of L is the graded $\mathbb{C}$-algebra

$$R(L) = \bigoplus_{m \geq 0} H^0(L^{\otimes m})$$

L is <u>finitely generated</u> if $R(L)$ is finitely generated as a $\mathbb{C}$-algebra.

<u>Basic example</u>: If $X = \mathbb{P}^n$, $L = \mathcal{O}(1)$, then $R(L) = \mathbb{C}[x_0, \dots, x_n]$.

<u>Recall</u>: It follows from semiample fibration theorem (+ some work) that if L is a globally generated line bundle on a normal variety X,

then $\exists$ an integer $m_0 > 0$ s.t. the multiplication maps

$$H^0(L^{\otimes a}) \otimes H^0(L^{\otimes b}) \to H^0(L^{\otimes (a+b)})$$

are surjective when $a, b \geq m_0$

In fact, one can show that for any coherent sheaf $\mathcal{F}$ on X,

$$H^0(\mathcal{F} \otimes L^{\otimes a}) \otimes H^0(L^{\otimes b}) \to H^0(\mathcal{F} \otimes L^{\otimes (a+b)})$$

is surjective for L globally generated and $a, b >> 0$.

This implies a thm of Zariski:

<u>Thm</u>: If L is semiample on a normal projective variety, then L is finitely generated.

<u>Pf</u>: Suppose $L^{\otimes k}$ is globally generated. Then for m sufficiently large,

write $m = (a+b)k + c$, $a, b \gg 0$

$$\Rightarrow H^0(L^{\otimes c} \otimes L^{\otimes a}) \otimes H^0(L^{\otimes b}) \twoheadrightarrow H^0(L^{\otimes m}). \square$$

A partial converse to the previous theorem also holds:

<u>Prop</u>: Let L be a finitely generated line bundle on a normal projective variety X. Then there's an integer $p > 0$, and a projective birational map $\nu : X' \longrightarrow X$ with X' normal and N' effective on X' s.t.

$$L' := \nu^*(L^{\otimes p}) \otimes \mathcal{O}_{X'}(-N')$$

is globally generated and $R(X, L)^{(p)} = R(X', L')$.

<u>Pf</u>: Exercise (see example 2.1.31 in Pos I).

<u>Rmk</u>: By the above prop, $\mathrm{vol}(L') = \mathrm{vol}(pL)$. L' is globally generated, so $\mathrm{vol}(L) = \frac{(L')^n}{p^n}$.

Thus, if L is finitely generated, $\mathrm{vol}(L)$ is rational.

$\boxed{\text{Apr 17}}$

<u>Zariski's construction</u>

<u>Goal</u>: Produce an example of a big and nef divisor D on a surface s.t. D is <u>not</u> finitely generated (and thus, in particular, not semiample).

Let $C_0 \subset \mathbb{P}^2$ be a smooth cubic curve,

and $\ell \subset \mathbb{P}^2$ the class of a line.

Choose $P_1, \dots, P_{12} \in C_0$ general so that

$\eta := \mathcal{O}_{C_0}(P_1 + \dots + P_{12} - 4\ell) \in \mathrm{Pic}^0(C_0)$ is non-torsion.

Let $X = \mathrm{Bl}_{\{P_1, \dots, P_{12}\}} \mathbb{P}^2$ and $\mu : X \longrightarrow \mathbb{P}^2$ the blowup,

and $E = \sum_{i=1}^{12} E_i$ the exceptional divisor, $H = \mu^* \ell$.

Let $\underset{C_0}{\overset{\cong}{C}} \equiv_{\mathrm{lin}} 3H - E$ be the strict transform of C_0.

Consider the divisor $D = 4H - E \equiv_{\mathrm{lin}} H + C$,

① <u>D is big + nef:</u>

H is big, C effective $\Rightarrow$ D is big

$\mathcal{O}_C(D) = \eta^*$, so $D \cdot C = 0$.

If $\underset{C}{\overset{\neq}{C'}} \subset X$ is another irred. curve, then H is nef, so

$D \cdot C' \geq 0$. $\Rightarrow$ D is big and nef.

② <u>$C \subseteq \mathrm{Bs}|mD|$</u>

$$0 \to \mathcal{O}_X(mD - C) \to \mathcal{O}_X(mD) \to \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \eta^{\otimes -m}}}{\mathcal{O}_C(mD)} \to 0$$

$\leftarrow$ No sections

so $H^0(\mathcal{O}_X(mD - C)) \cong H^0(\mathcal{O}_X(mD)) \Rightarrow C$ is in the base locus.

③ $|mD - C|$ is base point free

First of all, $D \equiv_{\text{lin}} H + C \Rightarrow D - C \equiv H$ is b.p.f.

Assume $(m-1)D - C$ is b.p.f.

Then $mD - C = (m-1)D + H$ and we have

$$0 \to \mathcal{O}_X(\underbrace{\overset{\alpha}{(m-1)D + H - C}}_{\text{bpf + bpf}}) \to \mathcal{O}_X(\overset{\beta}{(m-1)D + H}) \to \underbrace{\overset{\gamma}{\mathcal{O}_C((m-1)D + H)}}_{} \to 0$$

$\uparrow$ deg $\underset{2g+1}{\overset{\|}{3}}$ on $C \Rightarrow$ b.p.f.

Exercise: $H^1((m-1)D + H - C) = 0$. (induction)

Thus

$$\begin{array}{ccccccccc} 0 & \to & H^0(\alpha) & \to & H^0(\beta) & \to & H^0(\gamma) & \to & 0 \\ & & \twoheaddownarrow & & \downarrow & & \twoheaddownarrow & & \\ 0 & \to & \alpha & \longrightarrow & \beta & \longrightarrow & \gamma & \to & 0 \end{array}$$

So $|mD - C|$ is b.p.f.

④ $R(D)$ is not finitely generated

Claim: If $|mD|$ is finitely generated, then the multiplicity of a base curve must go to infinity as $m \to \infty$.

However $\mathrm{Bs}|mD| = C$, <u>with</u> multiplicity, so D is not finitely generated.

To prove the claim, we need a little more background:

Base loci of Big + Nef Divisors

<u>Wilson's Theorem</u>: Let X be an irreducible projective variety of dim n, D big and nef. Then $\exists$ $m_0 \in \mathbb{N}$ and N effective s.t.

$$|mD - N| \text{ is b.p.f. for } m \geq m_0$$

<u>Pf</u>: D big $\Rightarrow$ For any B v. ample, $\exists$ $m_0 > 0$ and effective N with

$$m_0 D \equiv_{lin} (n+1) B + N$$

So for $m \geq m_0$, $mD - N \equiv (m - m_0) D + (n+1) B \equiv B + nB + \text{nef}$

Thus, for $i = 1, \ldots, n$, we have

$$mD - N - iB \equiv kB + P \text{ with } k > 0, P \text{ nef}$$

So if we choose B to be a suff. large multiple of ample,

Fujita vanishing $\Rightarrow h^i(mD - N - iB) = 0 \quad \forall\ i > 0$

i.e. $mD - N$ is 0-regular w.r.t. B.

So, by a previous exercise, $|mD - N|$ is b.p.f. for $m \geq m_0$. $\square$

<u>Def</u>: Let $D \in \mathrm{Div}(X)$. Fix $|V| \subseteq |D|$ a linear subsystem and $x \in X$. The <u>multiplicity</u> of $|V|$ at x, $\mathrm{mult}_x |V|$ is the multiplicity at x of a general divisor in $|V|$.

In other words, $\mathrm{mult}_x |V| = \min_{D' \in |V|} \{\mathrm{mult}_x D'\}$

<u>Rmk</u>: $x \in \mathrm{Bs}|V| \iff \mathrm{mult}_x |V| > 0$.

<u>Cor</u>: Let D be big and nef. Then $\exists$ a constant $C > 0$ s.t. $\mathrm{mult}_x |mD| \leq C \quad \forall \; x \in X, \; m \in \mathbb{N}$. ($C$ doesn't depend on x, m)

[We'll see later that this can fail for D nef but not big.]

<u>Pf</u>: Wilson's thm $\Rightarrow \exists$ effective N s.t. $|mD - N|$ is b.p.f $\forall \; m > m_0$

So $\forall \; x \in X$, $\mathrm{mult}_x |mD - N| = 0$.

Choose $A \in |mD - N|$ s.t. $\mathrm{mult}_x A = 0$. Then

$\mathrm{mult}_x |mD| \leq \mathrm{mult}_x (A + N) = \mathrm{mult}_x N$, which is finite, and bounded above when varying x. $\square$

<u>Lemma</u>: Let X be normal, D a finitely generated divisor w/ $\kappa(D) \geq 0$. Then $\exists \; n \in \mathbb{N}(D)$ s.t.

$$\operatorname{mult}_x |knD| = k \cdot \operatorname{mult}_x |nD| \qquad \forall\ k \geq 1,\ x \in X.$$

<u>Pf:</u> Let $E \in |nD|$ s.t. $\operatorname{mult}_x |nD| = \operatorname{mult}_x E$.

Then for $k \geq 1$, $kE \in |knD|$

$$\Rightarrow \operatorname{mult}_x |knD| \leq \operatorname{mult}_x (kE) = k \operatorname{mult}_x E = k \cdot \operatorname{mult}_x |nD|$$

For the other inequality, D finitely generated $\Rightarrow$ there's some $n \in N(D)$ s.t. $H^0(\mathcal{O}_X(nD))$ generates $R(D)^{(n)}$ (can check)

So if $k > 0$, $H^0(knD)$ is a degree k polynomial in sections of $\mathcal{O}_X(nD)$. i.e. $H^0(nD)^{\otimes k} \longrightarrow H^0(knD)$ is surjective.

Notice: If $f, g \in H^0(knD)$, then $\operatorname{mult}_x (f+g) \geq \min\{\operatorname{mult}_x f, \operatorname{mult}_x g\}$.

Thus, for $E \in |knD|$, and each $x \in X$, there is some $E_x \in |knD|$ s.t. $E_x = \sum_{i=1}^{k} F_i$ with $F_i \in |nD|$ and

$$\operatorname{mult}_x E \geq \operatorname{mult}_x E_x \geq \sum \operatorname{mult}_x F_i \geq k \operatorname{mult}_x |nD|$$

$$\Rightarrow \operatorname{mult}_x |knD| \geq k \cdot \operatorname{mult}_x |nD| \text{ for each } x \in X,\ k \geq 1. \ \square$$

<u>Rmk:</u> Going back to Zariski's construction, take $x \in C$.

Then $\operatorname{mult}_x |knD| = \operatorname{mult}_x C$ for $k \geq 1$.

and $k \operatorname{mult}_x |nD| = k \cdot \operatorname{mult}_x C$. $\forall\ k \geq 1$.

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Thus, lemma implies that D is not finitely generated.

In fact...

Thm: Let X be normal, projective, and let $D \in \mathrm{Div}(X)$ be big and nef. Then D is finitely generated $\iff$ it's semiample.

Pf: We already know "$\Leftarrow$".

Assume D is finitely generated.

Then Lemma $\Rightarrow \exists\, n \in \mathbb{N}(D)$ s.t.

$$\mathrm{mult}_x |knD| = k\, \mathrm{mult}_x |nD|.$$

But LHS is bounded above $\forall\, k, x$, by corollary above.

$\Rightarrow \mathrm{mult}_x |nD| = 0 \quad \forall\, x$

$\Rightarrow |nD|$ is b.p.f. $\Rightarrow D$ is semiample. $\square$